II. SOLUTION BY A. M. HARDING, University of Arkansas.

The equation of the sphere is

$$\frac{x}{r} = \cos u \cos v, \qquad \frac{y}{r} = \cos u \sin v, \qquad \frac{z}{r} = \sin u.$$

Any plane through the center will be given by $ax + by + cz = 0$, or

$$a \cos u \cos v + b \cos u \sin v + c \sin u = 0.$$

Dividing by $\sqrt{a^2 + b^2}$, we obtain $\sin (v + \theta) + (c/\sqrt{a^2 + b^2}) \tan u = 0$, where θ is defined by the equations; $\sin \theta = a/\sqrt{a^2 + b^2}$, $\cos \theta = b/\sqrt{a^2 + b^2}$.

The transformation is

$$\frac{y}{r} = \operatorname{logtan}\left(\frac{u}{2} + \frac{\pi}{4}\right), \qquad \frac{x}{r} = v.$$

Solving the first of these equations for $\tan u$, we obtain

$$\tan u = \frac{e^{y/r} - e^{-y/r}}{2}.$$

Hence, by substitution, we have

$$-\frac{c}{\sqrt{a^2 + b^2}}(e^{y/r} - e^{-y/r}) = 2 \sin\left(\frac{x}{r} + \theta\right),$$

which is of the required form.

Also solved by PAUL CAPRON and the PROPOSER.

MECHANICS.

278. Proposed by A. M. HARDING, University of Arkansas.

A spherical shell of mass m explodes when moving with negligible velocity at a height of h feet above the ground. The shell is divided into very small particles, each of which moves, after the explosion, away from the center of the shell with a speed v, and ultimately falls to the ground. Find the total mass of the fragments which will be found per unit area at any specified distance from the point vertically underneath the shell.

SOLUTION BY H. S. UHLER, Yale University.

Let θ and φ denote the angles which the two trajectories, passing through the same point on the ground, make at the instant of the explosion, with the negative and positive directions of the axis of h respectively. φ must be acute but θ may be obtuse. The familiar equation $x = vt + \frac{1}{2}at^2$ leads to

$$r = \frac{v \sin \theta}{g}(+\sqrt{v^2 \cos^2 \theta + 2gh} - v \cos \theta), \tag{1}$$

$$r = \frac{v \sin \varphi}{g}(+\sqrt{v^2 \cos^2 \varphi + 2gh} + v \cos \varphi). \tag{2}$$

Rationalization of (1) and (2) gives

$$gr^2 + 2rv^2 \sin \theta \cos \theta - 2hv^2 \sin^2 \theta = 0, \tag{3}$$

$$gr^2 - 2rv^2 \sin \varphi \cos \varphi - 2hv^2 \sin^2 \varphi = 0. \tag{4}$$

Letting $\tau \equiv \tan\theta$ and $\gamma \equiv \cot\varphi$, (3) and (4) may be transformed to

$$(gr^2 - 2hv^2)\tau^2 + 2rv^2\tau + gr^2 = 0,$$

$$gr^2\gamma^2 - 2rv^2\gamma - (2hv^2 - gr^2) = 0.$$

Hence, under the conditions of the problem

$$\tau = \frac{r(v^2 + R)}{2hv^2 - gr^2}, \qquad \gamma = \frac{v^2 + R}{gr},$$

where

$$R \equiv + \sqrt{v^4 + 2ghv^2 - g^2r^2}.$$

Consequently

$$\cos\theta = \frac{2hv^2 - gr^2}{v\sqrt{2(2h^2v^2 - ghr^2 + r^2v^2 + r^2R)}}, \tag{5}$$

$$\cos\varphi = \frac{v^2 + R}{v\sqrt{2(gh + v^2 + R)}}, \tag{6}$$

By considering solid angles, it may be readily seen that the mass which is deposited on the circular area of radius r, corresponding to the cone of half-angle θ, is given by $\frac{1}{2}m(1 - \cos\theta) \equiv m_\theta$. Similarly the mass issuing from the upper or φ cone and falling on the same area equals $\frac{1}{2}m(1 - \cos\varphi) \equiv m_\phi$. The total mass on the area πr^2 is, therefore, $m_\theta + m_\phi \equiv m_r$. The total mass distributed over the ring $2\pi r dr$ is $(dm_r/dr)dr$, so that the surface density is given by

$$\sigma_r = \frac{1}{2\pi r}\left(\frac{dm_\theta}{dr} + \frac{dm_\phi}{dr}\right). \tag{7}$$

Assuming $v^4 + 2ghv^2 - g^2r^2 \mp 0$ the derivatives involved in (7) may be formed by the aid of (5) and (6), so that, in general

$$\sigma_r = \frac{m}{4\pi vR}\left\{\frac{(gr^2 + 2hR)(v^2 + R)^2}{[2(2h^2v^2 - ghr^2 + r^2v^2 + r^2R)]^{\frac{3}{2}}} + \frac{g^2(2gh + v^2 + R)}{[2(gh + v^2 + R)]^{\frac{3}{2}}}\right\}.$$

Special cases:

(*a*) When $R = 0$, that is, when r attains its superior limit $v/g\sqrt{v^2 + 2gh}$ the θ and φ trajectories coincide, the angles θ and φ become supplementary, and $\varphi = \tan^{-1}(1/v\sqrt{v^2 + 2gh})$.

(*b*) Taking $g = 32$ ft./sec.^{2}, $h = 5{,}000$ ft., $v = 1{,}024$ ft./sec., the maximum value of r equals $4{,}608\sqrt{66} \doteqdot 37{,}435.57$ ft. For $r = 0$, $\sigma = 3{,}669 \times 10^{-12}$m. When $2hv^2 - gr^2 = 0$ (*i. e.*, $\theta = \pi/2$) $\sigma = 1{,}612 \times 10^{-13}$ m. When $R = 6{,}604$, $\sigma = 1{,}823 \times 10^{-11}$ m. In the last instance $r \doteqdot 37{,}435.0001$ ft. If the entire mass were distributed uniformly over the maximum circle the surface density would be approximately $2{,}271 \times 10^{-13}$ m.

MECHANICS.

296. Proposed by C. N. SCHMALL, New York City.

A force F is exerted in moving a horizontal cylinder up an inclined plane by means of a crowbar of length l. If R be the radius of the cylinder, W its weight, φ the inclination of the plane to the horizontal and ψ the inclination of the crowbar to the horizon, show that

$$F = \frac{WR\sin\varphi}{l[1 + \cos(\varphi + \psi)]}.$$